STAT 685 Assignment 1 Due 9/10/98

Instructor: *Savas Papadopoulos*

1. Write a computer program that prints the inverse matrix, the determinant, the trace, the eigenvalues, and eigenvectors of the following covariance matrix. Use the same program to verify the spectral decomposition. Please submit the program and the output. SAS/ PROC IML is recommended for this class and help will be provided for it during office hours, but other packages such as S-PLUS or MATLAB are welcome to be used.

$$\mathbf{S} = \begin{pmatrix} 6.89 & & & \\ 6.25 & 15.58 & & \\ 5.84 & 5.84 & 10.76 & \\ 6.09 & 9.51 & 6.69 & 11.22 \end{pmatrix}$$

2. Let $\mathrm{x}_i \sim N(\mu, \Sigma)$ be independent random variables and

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i, \qquad \mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'.$$

The population and sample *augmented moments* are defined as

$$\Delta = \begin{pmatrix} \Sigma + \mu\mu' & \mu \\ \mu' & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \mathbf{S} + \bar{\mathbf{x}}\bar{\mathbf{x}}' & \bar{\mathbf{x}} \\ \bar{\mathbf{x}}' & 1 \end{pmatrix}, \quad \text{respectively.}$$

a) Let $\mathbf{y}_i = \begin{pmatrix} \mathbf{x}_i \\ 1 \end{pmatrix}$, verify that $\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}_i\mathbf{y}_i' = \mathbf{D}$ and $\mathrm{E}\{\mathbf{D}\} = \Delta$.

b) Show that if $\Delta > \mathbf{0}$, then $\Sigma > \mathbf{0}$.
Hint: Try to find a matrix $\mathbf{U}$ such that $\mathbf{U}'\mathbf{D}\mathbf{U} = \Sigma$.

c) Show that $|\Delta| = |\Sigma|$ and express Δ^{-1} in terms of μ and Σ^{-1}.

d) Show that $\log|\Sigma| + \mathrm{tr}(\mathbf{S}\Sigma^{-1}) + (\bar{\mathbf{x}} - \mu)'\Sigma^{-1}(\bar{\mathbf{x}} - \mu) + 1 = \log|\Delta| + \mathrm{tr}(\mathbf{D}\Delta^{-1})$.

Note that the above equation gives an alternative expression for the normal log-likelihood function. This way is useful in structural equation modeling when mean and covariance structures are desired to be fitted using only covariance structure theory and software. This is feasible as we will see later in this class if we use augmented moments instead of covariance matrices.

STAT 685 Assignment 2 Due 9/17/98

Instructor: *Savas Papadopoulos*

1. Let $\mathbf{x}_i \sim N_4(\mathbf{0}, \Sigma)$, $i = 1,\ldots,75$, be independent random variables with

$$\Sigma = \begin{pmatrix} 6.89 & & & \\ 6.25 & 15.58 & & \\ 5.84 & 5.84 & 10.76 & \\ 6.09 & 9.51 & 6.69 & 11.22 \end{pmatrix}, \quad \text{and let} \quad \mathbf{S} = \frac{1}{74}\sum_{\mathbf{i=1}}^{75}(\mathbf{x_i} - \overline{\mathbf{x}})(\mathbf{x_i} - \overline{\mathbf{x}})'.$$

Write a computer program that computes and prints:

(a) Var(s_{33})=

(b) Cov(s_{23}, s_{34})=

Please submit the program and the output.

2. For the following vectors and matrices, $\mathbf{x}$ nx1, $\mathbf{y}$ qx1, $\mathbf{A}$ nxn, and $\mathbf{B}$ pxq, show that

$$(\mathbf{x}' \otimes \mathbf{B})(\mathbf{A} \otimes \mathbf{y}) = \mathbf{Byx}'\mathbf{A}.$$

3. For the following nxn matrices, Σ, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ satisfying

$$\boldsymbol{\Sigma} = \mathbf{A}\boldsymbol{\Sigma}\mathbf{B}' + \mathbf{C}$$

show that

$$\text{vec}(\boldsymbol{\Sigma}) = [\mathbf{I} - (\mathbf{B} \otimes \mathbf{A})]^{-1}\text{vec}(\mathbf{C})$$

assuming that the inverse of $\mathbf{I} - (\mathbf{B} \otimes \mathbf{A})$ exists.

STAT 685 **Assignments 3-4** **Due 10/1/98**

Instructor: *Savas Papadopoulos*

1. Consider the regression model such that $y_j = \beta_0 + \beta_1 x_j + \varepsilon_j, \quad j = 1,2,\cdots,n$ assuming

$x_j \sim N(\mu_x, \sigma_{xx})$ and $\varepsilon_j \sim N(0, \sigma_{\varepsilon\varepsilon})$. Let $\mathbf{z}_j = (y_j \ x_j)'$ and $\theta = (\beta_0\ \beta_1\ \mu_x\ \sigma_{xx}\ \sigma_{\varepsilon\varepsilon})'$.

Then the mean and covariance structures for $\mathbf{z}_j$ are, respectively,

$$\mu(\theta) = \begin{pmatrix} \beta_0 + \beta_1\mu_x \\ \mu_x \end{pmatrix}, \text{ and } \Sigma(\theta) = \begin{pmatrix} \beta_1^2\sigma_{xx} + \sigma_{\varepsilon\varepsilon} & \beta_1\sigma_{xx} \\ \beta_1\sigma_{xx} & \sigma_{xx} \end{pmatrix}$$

i) Write explicitly the following matrices, $\mathbf{D}\Sigma(\theta)$, $H\Sigma(\theta)$, $\frac{\partial\Sigma(\theta)}{\partial\theta}$, $\frac{\partial\Sigma(\theta)}{\underline{\underline{\partial\theta}}}$, $\nabla\mu(\theta)$, and $D_{32}^2\mu_1((2\ \ \text{-}1\ 3\ 2\ 0.9)')$.

ii) Suppose that $x_j = (u_j + v_j)/2$ and let $\lambda = (\beta_0\ \beta_1\ \mu_u\ \mu_v\ \sigma_{uu}\ \sigma_{vv}\ \sigma_{uv}\ \sigma_{\varepsilon\varepsilon})'$.

Find $\mathbf{D}\Sigma(\lambda)$ by applying the chain rule for the functions $\Sigma(\theta)$ and $\theta(\lambda)$.

2. Let $\varphi(\mathbf{X}) = \text{tr}(\mathbf{AXBX}')$.

i) Find $\mathbf{d}\varphi(\mathbf{X})$, $\mathbf{d}^2\varphi(\mathbf{X})$, $\mathbf{D}\varphi(\mathbf{X})$, and $\frac{\partial\varphi(\mathbf{X})}{\partial\mathbf{X}}$.

ii) Show that $H\varphi(\mathbf{X}) = \mathbf{B}' \otimes \mathbf{A} + \mathbf{B} \otimes \mathbf{A}'$

3. Show that for **X,** an nxq matrix, holds that $\frac{\partial\mathbf{X}}{\underline{\underline{\partial\mathbf{X}}}} = \frac{\partial\mathbf{X}}{\partial\mathbf{X}} = \text{vec}(\mathbf{I}_n)\text{vec}'(\mathbf{I}_q)$

4. Show that for any vector function $\frac{\partial\mathbf{f}(\mathbf{x})}{\underline{\underline{\partial\mathbf{x}'}}} = \frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}'} = \mathbf{Df}(\mathbf{x})$.

STAT 685 Assignment 5 Due 10/22/98

Instructor: *Savas Papadopoulos*

1. Redo Exercise 1 of Assignment 2 in a different way. By hand now, compute the two quantities using the moments of the normal distributions given in class.

2. Consider the given example with the three measures of strength for 62 Alaskan earthquakes. The data set is available at "/home/savas/stat685/earthq.dat".

i) Apply Hausman's test, and report H_0, H_1, the test-statistic value, and the p-value. Use "PROC REG" with the "OUTPUT" option and submit the SAS program.

ii) Fit the following measurement error model with an instrumental variable w_j:

$$\begin{aligned} y_j &= \beta_0 + \beta_1 x_j^* + \varepsilon_j \\ x_j &= \qquad\quad x_j^* + u_j \qquad j = 1,2,\cdots,n \\ x_j^* &= \gamma_0 + \gamma_1 w_j + v_j \end{aligned}$$

Assume independence among the three errors. Report all the parameter estimates and their asymptotic standard errors (a.s.e.'s). Use "PROC CALIS" and submit the SAS program.

STAT 685 Assignment 6 Due 10/27/98

Instructor: *Savas Papadopoulos*

1. Verify the rank and order conditions for identification of the second behavioral equation in Klein's Model.

2. If $\mathbf{x}_i \sim N_p[\mu(\theta), \Sigma(\theta)]$, $i = 1, \ldots, n$, and are independent show that the MLE of θ can be obtained by minimizing the following function:

$$L(\theta) = \log|\Sigma| + \frac{(n-1)}{n}\mathrm{tr}(\mathbf{S}\Sigma^{-1}) + (\bar{\mathbf{x}} - \mu)'\Sigma^{-1}(\bar{\mathbf{x}} - \mu)$$

where **S** is the unbiased sample covariance matrix.

STAT 685 Assignment 7 Due 12/10/98

Instructor: *Savas Papadopoulos*

1. Read carefully the handout given in class discussing the nature of econometrics, etc. Write a short (close to 10 lines) and meaningful summary of it.

2. A random sample of 20 observations is generated for the x and y variables of the following system of simultaneous equations. The data set is available at /home/savas/stat685/sim_eq.dat, and consists of 8 columns corresponding to x_1-x_5 and y_1-y_3. Note that x_1=1 stands for the intercept.

$$y_1 = -60 - 10y_2 + 2.5y_3 + e_1$$
$$y_2 = 40 + 0.2y_1 - 4x_2 - 6x_3 + 1.5x_4 + e_2$$
$$y_3 = -10 + 2y_2 + 80x_2 + 5x_5 + e_3$$

with error covariance matrix $\Sigma = \begin{pmatrix} 227.55 & & \\ 8.91 & .66 & \\ -56.89 & -1.88 & 15.76 \end{pmatrix}$.

Use SAS with PROC SYSLIN and report estimates and estimated standard errors (within the parenthesis) for the intercepts and coefficients only. Obtain OLS, 2SLS, 3SLS, FIML, and LIML and report the results in the given space. Compare the different estimates to the true values and discuss their performance. Do the properties and comments in your notes for estimates and standard errors are supported from the results? Write explicitly which ones are supported and which ones are not?

i) For OLS:

$$y_1 = \underset{(\quad\quad)}{\underline{\quad\quad}} + \underset{(\quad\quad)}{\underline{\quad\quad}} y_2 + \underset{(\quad\quad)}{\underline{\quad\quad}} y_3 + e_1$$

$$y_2 = \underset{(\quad\quad)}{\underline{\quad\quad}} + \underset{(\quad\quad)}{\underline{\quad\quad}} y_1 + \underset{(\quad\quad)}{\underline{\quad\quad}} x_2 + \underset{(\quad\quad)}{\underline{\quad\quad}} x_3 + \underset{(\quad\quad)}{\underline{\quad\quad}} x_4 + e_2$$

$$y_3 = \underset{(\quad\quad)}{\underline{\quad\quad}} + \underset{(\quad\quad)}{\underline{\quad\quad}} y_2 + \underset{(\quad\quad)}{\underline{\quad\quad}} x_2 + \underset{(\quad\quad)}{\underline{\quad\quad}} x_5 + e_3$$

i) For 2SLS:

$$y_1 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} y_3 + e_1$$

$$y_2 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_1 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_3 + \frac{\qquad}{(\qquad)} x_4 + e_2$$

$$y_3 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_5 + e_3$$

i) For 3SLS:

$$y_1 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} y_3 + e_1$$

$$y_2 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_1 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_3 + \frac{\qquad}{(\qquad)} x_4 + e_2$$

$$y_3 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_5 + e_3$$

i) For FIML:

$$y_1 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} y_3 + e_1$$

$$y_2 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_1 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_3 + \frac{\qquad}{(\qquad)} x_4 + e_2$$

$$y_3 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_5 + e_3$$

i) For LIML:

$$y_1 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} y_3 + e_1$$

$$y_2 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_1 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_3 + \frac{\qquad}{(\qquad)} x_4 + e_2$$

$$y_3 = \frac{\qquad}{(\qquad)} + \frac{\qquad}{(\qquad)} y_2 + \frac{\qquad}{(\qquad)} x_2 + \frac{\qquad}{(\qquad)} x_5 + e_3$$

STAT 685 Midterm Due 10/29/98

(Take home)

Instructor: *Savas Papadopoulos*

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1. [45 points] For a pxp positive definite matrix **A** show that if (Error! **No text of specified style in document.**.5

$$\mathrm{tr}(\mathbf{A}) - \log|\mathbf{A}| - p = 0$$

then **A**=**I**.

2. [30 points] Solve the following matrix (for **X**,

$$\mathbf{AX} + \mathbf{XA} = \mathbf{B}$$

where **A** is nxn known positive definite, **B** is nxn known, and **X** is nxn unknown. That is, write **X** or a vector function of it in terms of **A**, **B**, and possibly other known matrices.

3. [75 points] For the function

$$\mathrm{L}(\theta) = \log|\Sigma(\theta)| + \mathrm{tr}[\mathbf{S}\Sigma^{-1}(\theta)] + [\bar{\mathbf{x}} - \mu(\theta)]'\Sigma^{-1}(\theta)[\bar{\mathbf{x}} - \mu(\theta)]$$

show that its Jacobian matrix can be expressed as

$$\mathbf{D}[\mathrm{L}(\theta)] = -2\begin{pmatrix} \bar{\mathbf{x}} - \mu(\theta) \\ \mathrm{v}\{\mathbf{S} - \Sigma(\theta) + [\bar{\mathbf{x}} - \mu(\theta)][\bar{\mathbf{x}} - \mu(\theta)]'\} \end{pmatrix}' \begin{pmatrix} \Sigma^{-1}(\theta) & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{D}_p'[\Sigma^{-1}(\theta) \otimes \Sigma^{-1}(\theta)]\mathbf{D}_p \end{pmatrix} \begin{pmatrix} \mathbf{D}[\mu(\theta)] \\ \mathbf{D}\{\mathrm{v}[\Sigma(\theta)]\} \end{pmatrix}$$

where $\mu(\theta)$ and $\bar{\mathbf{x}}$ are px1, and $\Sigma(\theta)$ is pxp.